Turk. J. Math. Comput. Sci.
5(2016) 1–7

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An Application of Improved Bernoulli Sub-Equation Function Method to The Nonlinear Time-Fractional Burgers Equation

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Received: 27-09-2016 • Accepted: 09-11-2016

Abstract. In this work, we study on the improved Bernoulli sub-equation function method. We apply this method to the nonlinear time-fractional Burgers equation. We obtain new analytical solutions to this model for values of n, m and M. Numerical simulation were depicted for different values of alpha.

2010 AMS Classification: XXXX

Keywords: Improved Bernoulli sub-equation function method, time-fractional Burgers equation, exponential function solution.

1. Introduction

Burgers equation which is popular model was firstly researched by J.M. Burgers in 1948. He worked at theory of turbulence [10]. Burgers equation is applied wide areas at mathematics, such as heat conduction, acoustic waves, modeling of dynamics [11,21,22]. The space and time- fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe [24]. Many researchers have studied on the numerical and analytical solutions of the space and time fractional Burgers equation.
The space and time-fractional Burgers equation was firstly studied by Momani via the Adomian decomposition method [20].In following years, differential transform method was applied time- and space- fractional generalized Burgers equation [19]. Mohyud-Din, S.T et al. have applied the homotopy analysis method to obtain the solutions of the space and time-fractional Burgers equations [25]. Harris, P.A et al. have studied on the analytic solutions of Burger type nonlinear fractional equations by invariant subspace method [16]. Many analytical methods have been presented for solving fractional differential equations, such as (G'/G) expansion method, fractional sub-equation method, exponential function method [9, 12, 14, 15, 26–28]. In this work, we have studied on the improved Bernoulli sub-equation function method for obtaining the new travelling wave solutions to the time-fractional Burgers equation.

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2. Preliminaries

In this section, we show the newly established definition of fractional differentiation.

Definition 2.1. Let $f \in H^1(a,b), a < b, \alpha \in [0,1]$ then, the definition of the new Caputo fractional derivative is [1, 2];

$$D_t^{\alpha}[f(t)] = \frac{M(\alpha)}{1-\alpha}\int_a^t f'(x)exp\left(-\alpha\frac{t-x}{1-\alpha}\right)dx,$$

where $M(\alpha)$ denotes a normalization function obeying $M(0) = M(1) = 1$. However, if the function does not belong to $H^1(a,b)$ then, the derivative has the form:

$$D_t^{\alpha}[f(t)] = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^t [f(t)-f(x)]exp\left(-\alpha\frac{t-x}{1-\alpha}\right)dx. \tag{2.1}$$

If $\sigma = (1-\alpha)/\alpha \in [0,\infty], \alpha = 1/(1+\sigma) \in [0,1]$ then Eq. (2.1) assumes the form:

$$D_t^{\sigma}[f(t)] = \frac{N(\sigma)}{\sigma}\int_a^t f'(x)exp\left(-\frac{t-x}{\sigma}\right)dx, N(0) = N(\infty) = 1.$$

Definition 2.2 (Atangana-Baleanu fractional derivative in Riemann-Liouville sense)**.** Let $f \in H^1(a,b), a < b, \alpha \in [0,1]$ and not necessary differentiable then, the definition of the new fractional derivative is given as [2]:

$${}^{ABR}_{a}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right].$$

Definition 2.3 (Atangana-Baleanu fractional derivative in Caputo sense)**.** Let $f \in H^1(a,b), a < b, \alpha \in [0,1]$ then, the definition of the new fractional derivative is given as [3]:

$${}^{ABC}_{a}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right].$$

The B has the same properties as in Caputo and Fabrizio case.

Definition 2.4 (Atangana-Baleanu fractional integral)**.** The fractional integral associate to the new fractional derivative with non-local kernel is defined as:

$${}^{AB}_{a}I_t^{\alpha}[f(t)] = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(j)(t-j)^{\alpha-1}dj.$$

When alpha is zero we recover the initial function and if also alpha is 1, we obtain the ordinary integral [2].

3. The Improved Bernoulli Sub-Equation Function Method (IBSEFM)

Improved Bernoulli sub-equation function method (IBSEFM) formed by modifying the Bernoulli sub-equation function method [13] will be given in this sub-section. We consider the following four steps.

Step 1. Let's consider the following fractional differential equation;

$$P(D_t^{\alpha}u, u_x, u_t, u_{xt}, ...) = 0, \tag{3.1}$$

and take the wave transformation;

$$u(x,t) = U(\eta), \quad \eta = kx - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{3.2}$$

where k,c are constants and can be determined later. Using the chain rule as following;

$$D_t^{\alpha}u = \sigma_t'\frac{dU}{d\eta}D_t^{\alpha}\eta, \tag{3.3}$$

where σ_t' is defined as the sigma fractal indexes [4–8, 17, 18, 23] without loss of generality, we can take $\sigma_t' = \omega$ in which ω is a constant. Substituting Eq. (3.2) and Eq. (3.3) in Eq. (3.1), we obtain the following nonlinear ordinary differential equation;

$$N(U, U', U'', U''', ...) = 0. \tag{3.4}$$

Step 2. Considering trial equation of solution in Eq. (3.4), it can be written as following;

$$U(\eta) = \frac{\sum_{i=0}^{n} a_i F^i(\eta)}{\sum_{j=0}^{m} b_i F^j(\eta)} = \frac{a_0 + a_1 F(\eta) + a_2 F^2(\eta) + ... + a_n F^n(\eta)}{b_0 + b_1 F(\eta) + b_2 F^2(\eta) + ... + b_m F^m(\eta)}. \tag{3.5}$$

According to the Bernoulli theory, we can consider the general form of Bernoulli differential equation for F' as following;

$$F' = bF + dF^M, \quad b \neq 0, \quad d \neq 0, \quad M \in \mathbb{R} - \{0, 1, 2\}, \tag{3.6}$$

where $F = F(\eta)$ is Bernoulli differential polynomial. Substituting above relations in Eq. (3.4), it yields us an equation of polynomial $\Omega(F)$ of$F$ as following;

$$\Omega(F) = \rho_s F^s + ... + \rho_1 F + \rho_0 = 0.$$

According to the balance principle, we can determine the relationship between n, m and M.

Step 3. The coefficients of $\Omega(F)$ all be zero will yield us an algebraic system of equations;

$$\rho_i = 0, \quad i = 0, ..., s.$$

Solving this system, we will specify the values of $a_0, a_1, ..., a_n$ and $b_0, b_1, ..., b_m$.

Step 4. When we solve nonlinear Bernoulli differential equation Eq. (3.6), we obtain the following two situations according to b and d,

$$F(\eta) = \left[\frac{-d}{b} + \frac{E}{e^{b(M-1)\eta}}\right]^{\frac{1}{1-M}}, \quad b \neq d, \tag{3.7}$$

$$F(\eta) = \left[\frac{(E-1) + (E+1)tanh(b(1-M)\frac{\eta}{2})}{1 - tanh(b(1-M)\frac{\eta}{2})}\right], \quad b = d, \quad E \in \mathbb{R}.$$

Using a complete discrimination system for polynomial of $F(\eta)$, we obtain the analytical solutions to the Eq. (3.4) with the help of Mathematica 9 and classify the exact solutions to Eq. (3.4). For a better interpretations of results obtained in this way, we can plot two- and three- dimensional figures of analytical solutions by considering suitable values of parameters.

4. Application of Approach Improved

In this subsection, we have obtained some prototype analytical solutions to the nonlinear time-fractional Burgers equation as following.

Example: We consider the following time fractional Burgers equation [9];

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{4.1}$$

where α is a order of time fractional Burgers equation. We apply the following transformations;

$$u(x,t) = U(\eta),$$
$$\eta = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{4.2}$$

where k and c are constants. By substituting Eq. (4.2) into Eq. (4.1) we have ordinary differential equation;

$$-c\omega U' + k\varepsilon U U' - k^2 \nu U'' = 0, \tag{4.3}$$

where $U' = \frac{dU}{d\eta}$, $\omega \neq 1$. By integrating once, we have;

$$\xi_0 - c\omega U + k\varepsilon U \frac{U^2}{2} - k^2 \nu U' = 0, \tag{4.4}$$

where ξ_0 is an integration constant. When we reconsider to Eq. (4.3) and Eq. (4.4) for balance principle, we obtain following relationship for n and M;

$$M + m = n + 1.$$

This resolution procedure is performed and we can obtain some analytical solutions as follows.

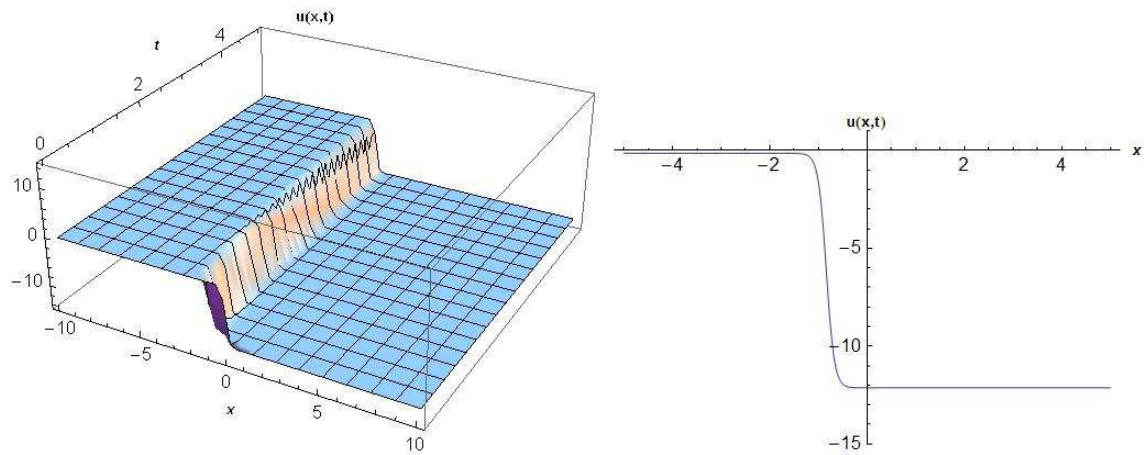

FIGURE 1. The 3D and 2D surfaces of the analytical solution Eq. (4.8) by considering the values $b = -2$, $d = 0.7$, $k = 3$, $\alpha = 0.19$, $\nu = 0.1$, $\varepsilon = 0.2$, $\xi = 0.5$, $E = 2$, $\omega = 0.8$, $b_0 = 0.7$ $-10 < x < 10$, $0 < t < 5$ for 3D graphics and $t = 0.01$ for 2D surfaces.

Case 1: If we take $M = 3, n = 4$ and $m = 2$ for Eq. (3.5) and Eq. (3.6), then, we can write following equations;

$$U = \frac{a_0 + a_1F + a_2F^2 + a_3F^3 + a_4F^4}{b_0 + b_1F + b_2F^2} = \frac{\Upsilon}{\Psi} \tag{4.5}$$

and

$$U' = \frac{\Upsilon'\Psi - \Upsilon\Psi'}{\Psi^2}, \tag{4.6}$$

where $F' = bF + dF^3, a_4 \neq 0, b_2 \neq 0$. When we use Eq. (4.5) and Eq. (4.6) in Eq. (4.4), we get a system of algebraic equations. Solving this algebraic system of equations with the help of Wolfram Mathematica 9 yields the following coefficients,

$$a_0 = \frac{2bk^2\nu b_0 - \sqrt{2k(2b^2k^3\nu^2 + \varepsilon\xi)b_0^2}}{k\varepsilon}, \quad a_1 = \frac{\left(2bk^2\nu - \sqrt{2k(2b^2k^3\nu^2 + \varepsilon\xi)}\right)b_1}{k\varepsilon},$$
$$a_3 = \frac{4dk\nu b_1}{\varepsilon}, \quad a_2 = \frac{4dk\nu b_0 + 2bk\nu b_2 - \sqrt{2k\,(2b^2k^3\nu^2 + \varepsilon\xi)}b_2}{k\varepsilon}, \quad a_4 = \frac{4dk\nu b_2}{\varepsilon},$$
$$c = -\frac{\sqrt{2}}{\omega}\sqrt{k\,(2b^2k^3\nu^2 + \varepsilon\xi)}. \tag{4.7}$$

Substituting Eq. (4.7) coefficients along with Eq. (3.7) in Eq. (4.5), we obtain following solution of the time fractional Burgers equation;

$$u_1(x,t) = -\frac{\sqrt{2}b_0}{k\varepsilon}\sqrt{k\,(2b^2k^3\nu^2 + \varepsilon\xi)} + \frac{2k\nu}{\varepsilon}\left(b + \frac{2d}{-\frac{d}{b} + e^{-2b\left(kx + \frac{t^\alpha\sqrt{2k\left(2b^2k^3\nu^2+\varepsilon\xi\right)}}{\omega\Gamma[1+\alpha]}\right)}E}\right), \tag{4.8}$$

where $b, b_0, \varepsilon, \xi, d, \alpha, k, \nu, \omega, e, E$ are constants and not zero.

Case 2: If we take $M = 3, n = 4$ and $m = 2$ for Eq. (3.5) and Eq. (3.6), and also, use Eq. (4.5) and Eq. (4.6) in Eq. (4.4), we get a system of algebraic equations. By solving this algebraic system of equations with the help of Wolfram Mathematica 9, it yields the following coefficients.

$$a_1 = \frac{\left(2bk^2\nu + \sqrt{2k(2b^2k^3\nu^2 + \varepsilon\xi)}\right)b_1}{k\varepsilon}, \quad b_0 = \frac{-2bk^2\nu a_0 + \sqrt{2k(2b^2k^3\nu^2 + \varepsilon\xi)}}{2\varepsilon},$$

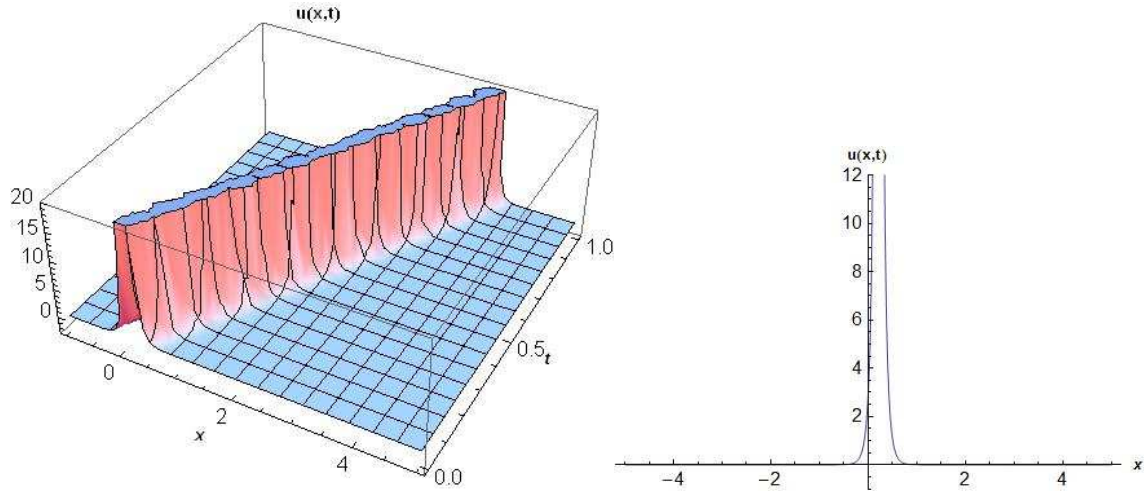

FIGURE 2. The 3D and 2D surfaces of the analytical solution Eq. (4.10) by considering the values $b = 2$, $d = 0.7$, $k = 3$, $\alpha = 0.99$, $\nu = 0.1$, $\varepsilon = 0.3$, $\xi = 0.4$, $E = 3$, $\omega = 0.38$, $a_0 = 0.3$ $-1 < x < 5$, $0 < t < 1$ for 3D graphics and $t = 0.01$ for 2D surfaces.

$$a_4 = \frac{4dk\nu b_2}{\varepsilon}, \quad a_3 = \frac{4dk\nu b_1}{\varepsilon}, \quad c = \frac{\sqrt{2k\left(2b^2k^3\nu^2 + \varepsilon\xi\right)a_0^2}}{\omega a_0},$$

$$a_2 = \frac{\left(-4bdk^4\nu^2 a_0 + \xi\sqrt{2k(2b^2k^3\nu^2 + \varepsilon\xi)b_2} + 2k^2\nu\left(\sqrt{2}d\sqrt{k\left(2b^2k^3\nu^2 + \varepsilon\xi\right)a_0^2} + b\xi b_2\right)\right)}{k\varepsilon\xi}. \tag{4.9}$$

Substituting Eq. (4.9) coefficients along with Eq. (3.7) in Eq. (4.5), we obtain following solution of the time fractional Burgers equation;

$$u_2(x,t) = \frac{8E\nu b^3 dk}{\varepsilon} e^{2b\left(kx + \frac{t^\alpha\sqrt{2k\left(2b^2k^3\nu^2+\varepsilon\xi\right)}}{\omega\Gamma[1+\alpha]}\right)}\left(de^{2bkx} - be^{\left(\frac{2bt^\alpha\sqrt{2k\left(2b^2k^3\nu^2+\varepsilon\xi\right)}}{\omega\Gamma[1+\alpha]}\right)}{}_E\right)^{-2}, \tag{4.10}$$

where $b, d, k, \varepsilon, \xi, \alpha, \nu, \omega, a_0, e, E$ are constants and not zero.

Case 3: If we take $M = 3, n = 3$ and $m = 1$ for Eq. (3.5) and Eq. (3.6), then, we can write following equations;

$$U = \frac{a_0 + a_1F + a_2F^2 + a_3F^3}{b_0 + b_1F} = \frac{\Upsilon}{\Psi}, \tag{4.11}$$

and

$$U' = \frac{\Upsilon'\Psi - \Upsilon\Psi'}{\Psi^2}, \tag{4.12}$$

where $F' = bF + dF^3, a_3 \neq 0, b_1 \neq 0$. When we use Eq. (4.11) and Eq. (4.12) in Eq. (4.4), we get a system of algebraic equations. Solving this algebraic system of equations with the help of Wolfram Mathematica 9 yields the following coefficients,

$$a_1 = \frac{\left(2bk^2\nu a_0 + \sqrt{2}\sqrt{k(2b^2k^3\nu^2 + \varepsilon\xi)a_0^2}\right)b_1}{k\varepsilon a_0}, \quad a_3 = \frac{4dk\nu b_1}{\varepsilon}, c = \frac{\sqrt{2}}{\omega a_0}\sqrt{k\left(2b^2k^3\nu^2 + \varepsilon\xi\right)a_0^2},$$

$$b_0 = \frac{-2bk^2\nu a_0 + \sqrt{2}\sqrt{k(2b^2k^3\nu^2 + \varepsilon\xi)a_0^2}}{2\xi}, \quad a_2 = \frac{2dk\nu\left(-2bk^2\nu a_0 + \sqrt{2}\sqrt{k(2b^2k^3\nu^2 + \varepsilon\xi)a_0^2}\right)b_1}{\varepsilon\xi_0}. \tag{4.13}$$

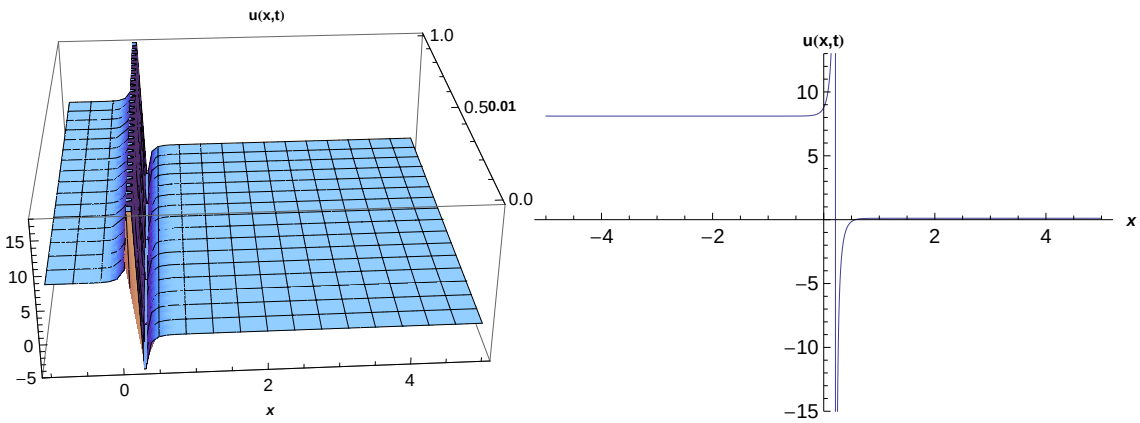

FIGURE 3. The 3D and 2D surfaces of the analytical solution Eq. (4.14) by considering the values $b = 2$, $d = 0.7$, $k = 3$, $\alpha = 0.99$, $\nu = 0.1$, $\varepsilon = 0.3$, $\xi = 0.4$, $E = 3$, $\omega = 0.38$, $a_0 = 0.3$ $-1 < x < 5, 0 < t < 1$ for 3D graphics and $t = 0.01$ for 2D surfaces.

If we use the Eq. (4.13) coefficients along with Eq. (3.7) in Eq. (4.11), we obtain following solution of the time fractional Burgers equation;

$$u_3(x,t) = \frac{2k^2\nu}{k\varepsilon}\left(b + \frac{2d}{-\frac{d}{b} + Ee^{\left(-2bkx + \frac{2\sqrt{2}bt^{\alpha}\sqrt{k\left(2b^2k^3\nu^2+\varepsilon\xi\right)a_0^2}}{\omega\Gamma[1+\alpha]a_0}\right)}}\right) + \frac{\sqrt{2}\sqrt{k\left(2b^2k^3\nu^2 + \varepsilon\xi\right)a_0^2}}{k\varepsilon a_0}. \tag{4.14}$$

where $b, d, k, \varepsilon, \xi, \alpha, \nu, \omega, a_0, e, E$ are constants and not zero.

5. CONCLUSIONS

In this manuscript, we have successfully applied the IBSEFM to the nonlinear time-fractional Burgers equation for obtaining some new travelling wave solutions. Then, we have obtained some new travelling wave solutions of Eq. (4.1). It has been observed that all analytical solutions obtained in this paper verify the nonlinear ordinary differential equation Eq. (4.4) which is obtained from nonlinear time-fractional Burgers equation under the terms of wave transformation relationship. All necessary computational calculations and Figure 1 , Figure 2 and Figure 3 have been obtained by using Wolfram Mathematica 9 version.

According to the figures, it one can see that the formats of travelling wave solutions in two- and three-dimensional surfaces are similar to the physical meaning of results. If we take more values of coefficients, we can obtain more travelling wave solutions for this model. This method is very reliable and efficient and submits new travelling wave solutions to literature beforehand. Thus, we think that this method can be applied to other nonlinear fractional differential models.

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